5.1 Discrete Bivariate and Multivariate Distributions

Often times we are dealing with more than a single measurement (random variable) at one time. Some examples:

* We randomly choose an individual and measure and .
* We randomly choose a patient and measure ,  and .
* Three individuals each toss 10 coins and ,  and 

In each case above, two or more values are being randomly selected jointly. We desire to look at the distribution of pairs (or triples in case 3) of values that can be randomly chosen. We are looking at the discrete case first because it is much easier to visualize than the continuous case.

Let X and Y be two discrete random variables that occur jointly. That is there is a single random action that produces two values, x and y. We define the joint pmf for X and Y as . This is called the joint pmf because it answers questions that are asked “jointly” about X and Y.

As in the case with a single random variable, the pmf can be given in table format or in functional format. Throughout this section and the next, we will be using the three joint pmfs listed below.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 | 4 |
|  | 0 | 0.01 | 0.05 | 0.06 | 0.20 | 0.07 |
| Y | 1 | 0.06 | 0.02 | 0.11 | 0.07 | 0.12 |
|  | 2 | 0.08 | 0.03 | 0.04 | 0.05 | 0.03 |

Example A:

Example B:  for 

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | Z=2 |  |  |  |  |  |  |  | Z=4 |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  | X |  |  |  |  |  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 |  |  |  |  | 0 | 1 | 2 | 3 |
|  | 1 | .01 | .02 | .03 | **.04** |  |  |  | 1 | .03 | .05 | .04 | .01 |
| Y | 2 | .02 | .03 | .04 | .05 |  |  | Y | 2 | .02 | .04 | .06 | .08 |
|  | 3 | .01 | .01 | .02 | .02 |  |  |  | 3 | .03 | **.07** | .02 | .08 |
|  | 4 | .03 | .02 | .01 | .01 |  |  |  | 4 | .01 | .02 | .03 | .04 |

Example C:

Example D: The joint pmf of the discrete random variables *X, Y*, and *Z* is given by

 for 

Definition: The support of the joint random variables X and Y is the ordered pairs  such that . The joint is defined similarly for more than two jointly distributed random variables.

**Example A1:** Determine . Instead of using the intersection symbol, we will ask this question as follows: Determine . We see from the table that 

We see that we can answer any joint probability question with no effort just by using the chart. Our functional notation for the joint pmf is . We can use the subscript in our notation when it seems useful to clarify the random variables in use.

**Example A2:** Determine . 

**Example A3:** Determine  and .

**Example B1:** Determine .  

**Example B2:** Determine  and .

**Example C1:** Determine . 

**Example C2:** Determine 

**Example D1:** 

**Example D2:** Determine .

**Example:** Determine the value of c that makes the table below a joint pmf.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 | 4 |
|  | 0 | c | 0.05 | 0.06 | 0.12 | 0.07 |
| Y | 1 | 0.06 | 0.02 | 0.03 | 0.07 | 0.12 |
|  | 2 | 0.08 | 0.03 | 0.04 | 0.05 | 0.03 |

The probabilities need to add to 1, so 

**Example:** Determine the value of c that makes the function below a joint pmf.

 for 

3(2)(1) + 3(2)(2) + 3(2)(3) + 3(1) + 3(4) + 3(9) = 78

Currently, the need for the subscripts seems unclear. That will change in the next section.

5.2 Discrete Marginal Distributions

Often times we have a joint pmf that allows us to answer any joint question about X and Y. We now look at answering questions about just X or just Y. These questions require the marginal distributions.

Definition: Given random variables X and Y with joint pmf , we define the 2 marginal distributions by

 for all  in the support of X,Y.

 for all  in the support of X,Y.

Things are so much simpler than they seem. Suppose that we wish to ask a question just about X or just about Y. We would use common sense (The Law of Total Probability) to answer the question.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 | 4 |
|  | 0 | 0.01 | 0.05 | 0.06 | 0.20 | 0.07 |
| Y | 1 | 0.06 | 0.02 | 0.11 | 0.07 | 0.12 |
|  | 2 | 0.08 | 0.03 | 0.04 | 0.05 | 0.03 |

**Example A4:**

Determine . To answer this we would do what we always do: add up all of the ways . We have exactly three ways where : . Thus,



We now have  and can determine the rest of  in a similar manner.





 and 

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 | 4 |
|  | 0 | 0.01 | 0.05 | 0.06 | 0.20 | 0.07 |
| Y | 1 | 0.06 | 0.02 | 0.11 | 0.07 | 0.12 |
|  | 2 | 0.08 | 0.03 | 0.04 | 0.05 | 0.03 |
|  |  | 0.15 | 0.10 | 0.21 | 0.32 | 0.22 |

We have written the story of X in the margin of the table. This is why we name this the marginal distribution of X. We determine the marginal distribution by summing (or collapsing) the other variable(s).

|  |  |  |
| --- | --- | --- |
|  |  |  |
| 0 | .15 |  |
| 1 | .10 |  |
| 2 | .21 |  |
| 3 | .32 |  |
| 4 | .22 |  |

If desired, we can display our marginal distributions in the usual way:

|  |  |
| --- | --- |
|  |  |
| 0 | .39 |
| 1 | .38 |
| 2 | .23 |

**Example A5:** Determine(You).

**Example B3:** Determine.  for 

We can verify that  is a pmf by making sure that the total probability is 1: 

**Example B4:** Determine. Summing the y-values gives: 

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | Z=2 |  |  |  |  |  |  |  | Z=4 |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  | X |  |  |  |  |  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 |  |  |  |  | 0 | 1 | 2 | 3 |
|  | 1 | .01 | .02 | .03 | .04 |  |  |  | 1 | .03 | .05 | .04 | .01 |
| Y | 2 | .02 | .03 | .04 | .05 |  |  | Y | 2 | .02 | .04 | .06 | .08 |
|  | 3 | .01 | .01 | .02 | .02 |  |  |  | 3 | .03 | .07 | .02 | .08 |
|  | 4 | .03 | .02 | .01 | .01 |  |  |  | 4 | .01 | .02 | .03 | .04 |

**Example C3:**

The joint distribution of X, Y and Z is given above. There are six different marginal distributions that can be derived from the joint. They are:



Determine . To determine this distribution, we need to sum over the variable Z for all combinations of x and y. That is, add up all of the ways that  for all combinations of *x* and *y*. For example, .

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 |
|  | 1 | .04 | .07 | .07 | .05 |
| Y | 2 | .04 | .07 | .10 | .13 |
|  | 3 | .04 | .08 | .04 | .10 |
|  | 4 | .04 | .04 | .04 | .05 |

We do this for each  combination and get the joint distribution for X and Y.



To determine the marginal for X or the marginal for Y, , we simply collapse out the other variable.

|  |  |
| --- | --- |
| x |  |
| 0 | .16 |
| 1 | .26 |
| 2 | .25 |
| 3 | .33 |

**Example C4:** Determine the marginal distribution . We simply need to sum out the variable Y (or collapse down).

**Example D3:** The joint pmf of the discrete random variables *X, Y*, and *Z* is given by

 for 

Determine . To determine this marginal distribution, we collapse (sum) out the z-variable. 

**Example D4:** Determine  and .

. 

**Example D5:** Determine 



|  |  |
| --- | --- |
| y |  |
| 1 | 15/56 |
| 2 | 18/56 |
| 3 | 23/56 |

Or, we can put in tabular form.

|  |  |
| --- | --- |
| x |  |
| 1 | 53/168 |
| 2 | 56/168 |
| 3 | 59/168 |

**Example D6:** Determine 



|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
|  |  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 | 4 |
|  | 0 | 0.01 | 0.05 | 0.06 | 0.20 | 0.07 |
| Y | 1 | 0.06 | 0.02 | 0.11 | 0.07 | 0.12 |
|  | 2 | 0.08 | 0.03 | 0.04 | 0.05 | 0.03 |

5.3 Discrete Conditional Distributions

**Example A6:** One question that might be asked is to determine . Our pmf notation for such a question would be .

To answer the question, we use portion of the whole. Since we are told that , the total probability has been reduced to .38. Of that probability of .38, .11 occurs when . So . We could also solve this problem using our definition of conditional probability. 

**Example A7:** We might be interested in the entire distribution. Using the above idea and notation we have



Since our problem originated in tabular form, we will put  in tabular form also. In our next example, the joint distribution will be in functional form so our marginal conditional distributions will be also.

|  |  |  |
| --- | --- | --- |
|  | x |  |
|  | 0 | .06/.38 |
|  | 1 | .02/.38 |
|  | 2 | .11/.38 |
|  | 3 | .07/.38 |
|  | 4 | .12/.38 |

|  |  |  |
| --- | --- | --- |
|  | x |  |
|  | 0 | .08/.23 |
|  | 1 | .03/.23 |
|  | 2 | .04/.23 |
|  | 3 | .05/.23 |
|  | 4 | .03/.23 |

**Example B5:**  for . Determine .

 with support .

**Example B6:**  for . Determine .

 with support .

**Example D7:** The joint pmf of the discrete random variables *X, Y*, and *Z* is given by

 for 

Determine . Using our definition of conditional probability, we would have . (We found  earlier.)

Notice that this looks like a function of x, y and z. However, if we specify a value for x and y, we would truly have a function of just z.

**Example D8:** . We could put this in table format.

|  |  |
| --- | --- |
| z |  |
| 1 | 8/57 |
| 2 | 15/57 |
| 3 | 34/57 |



**Example D9:** Determine . To do this we would need (according to our definition of conditional probability) to determine  and . These were done in D3 and D5. 

|  |  |
| --- | --- |
| x |  |
| 1 | 22/69 |
| 2 | 23/69 |
| 3 | 24/69 |

**Example D10:** Determine . .

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  |  |  | Z=2 |  |  |  |  |  |  |  | Z=4 |  |  |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
|  |  |  | X |  |  |  |  |  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 |  |  |  |  | 0 | 1 | 2 | 3 |
|  | 1 | .01 | .02 | .03 | .04 |  |  |  | 1 | .03 | .05 | .04 | .01 |
| Y | 2 | .02 | .03 | .04 | .05 |  |  | Y | 2 | .02 | .04 | .06 | .08 |
|  | 3 | .01 | .01 | .02 | .02 |  |  |  | 3 | .03 | .07 | .02 | .08 |
|  | 4 | .03 | .02 | .01 | .01 |  |  |  | 4 | .01 | .02 | .03 | .04 |

**Example C5:**

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  |  |  | X |  |  |
|  |  | 0 | 1 | 2 | 3 |
|  | 1 | .04 | .07 | .07 | .05 |
| Y | 2 | .04 | .07 | .10 | .13 |
|  | 3 | .04 | .08 | .04 | .10 |
|  | 4 | .04 | .04 | .04 | .05 |

Determine . In C3, we determined :

Here, we are given that . The total probability has been reduced to .26.

|  |  |
| --- | --- |
| x |  |
| 0 | .04/.26 |
| 1 | .08/.26 |
| 2 | .04/.26 |
| 3 | .10/.26 |

So, using portion of the whole, the table for  is:

|  |  |
| --- | --- |
| x |  |
| 0 | .16 |
| 1 | .26 |
| 2 | .25 |
| 3 | .33 |

We see that is very different from .